B. Sc. IIIrd Year Physics Paper- XV

Ch. 1 Classical Mechanics

1. Mechanics of Particle: Mechanics is the study of the motion of physical bodies. We have to study conservation laws for a particle in motion using Newtonian mechanics. The word "conservation" applies in the sense of constantness, when some characteristics of the motion of a system remain constant in time. There are conservation laws relating to energy, linear momentum, angular momentum, charge and various other quantities.

i) **Conservation of linear momentum:** Linear momentum is the product of mass of particle m and velocity v.

Linear momentum $P = mv$

Let F be force acting on a particle of mass m, then according to Newton's second law of motion,

Force = rate of change of momentum

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt}$$

If the total force acting on particle is zero i.e. F=0 then

$$\frac{dP}{dt} = 0$$

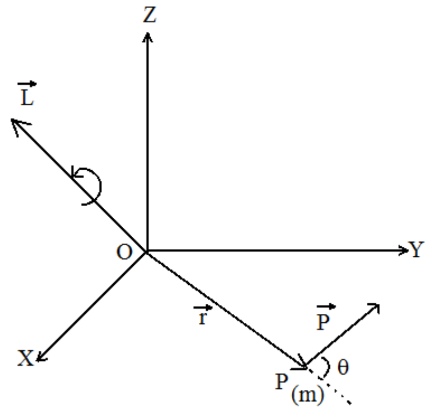
$$\therefore P = \text{constant}$$

$$\therefore P = mv = constant$$

Thus in the absence of external force, the linear momentum of a particle is constant i.e. linear momentum is conserved.

ii) **Conservation of angular momentum:** Angular momentum is the analogue of the linear momentum in case of rotational motion. It is the moment of linear momentum.

Consider a particle of mass m and linear momentum P at the position $\vec{r}$ relative to origin O as shown in figure.



The angular momentum $\vec{L}$ of particle with respect to origin O is defined as,

$$\vec{L} = \vec{r} \times \vec{P} \text{ -------------------- (1)}$$

Let F be force acting on particle then moment of force or torque about the origin O is given by,

$$\tau = \vec{r} \times \vec{P} \text{ -------------------- (2)}$$

Differentiate eq^n(1) with respect to *t*

$$\frac{dL}{dt} = \frac{d}{dt}(\vec{r} \times \vec{P})$$

$$= \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P}$$

But $\frac{d\vec{r}}{dt} \times \vec{P} = \vec{V} \times m\vec{V} = 0$

$$\therefore \frac{dL}{dt} = \vec{r} \times \frac{d\vec{P}}{dt}$$

But $\vec{F} = \frac{d\vec{P}}{dt}$

$$\therefore \frac{dL}{dt} = \vec{r} \times \vec{F} = \tau$$

$$\therefore \tau = \frac{dL}{dt}$$

If total torque acting on a particle is zero i.e. $\tau = 0$,

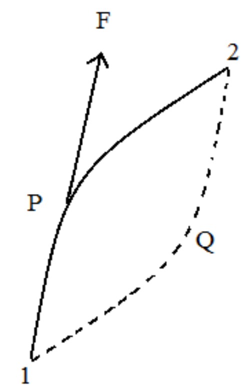
$$\frac{dL}{dt} = 0$$

$$L = constant$$

Thus in the absence of external torque, total angular momentum remains constant i.e. angular momentum is conserved.

iii) **Conservation of Energy:**

Suppose, under the action of a force the particle moves from position 1 to position 2.



Then work done by the particle is,

$$W_{12} = \int_1^2 F.dr \quad \text{------------------} \ (1)$$

According to Newton's second law of motion,

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt}$$

$$\therefore F.dr = m\frac{dv}{dt}.dr$$

$$= m\frac{dv}{dt}.\frac{dr}{dt}.dt = m.\frac{dv}{dt}.v.dt$$

Where $v = dr/dt$ is velocity of particle.

$$\therefore F.dr = \frac{d}{dt}\left[\frac{1}{2}mv^2\right]dt$$

From eq^n (1)

Work done by the particle is,

$$W_{12} = \int_1^2 F.dr = \int_1^2 \frac{d}{dt}\left[\frac{1}{2}mv^2\right]dt$$

$$= \left[\frac{1}{2}mv^2\right]_1^2 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

But $\frac{1}{2}mv^2 = T$ is kinetic energy of particle.

$$W_{12} = T_2 - T_1 \quad \text{------------------------} \ (2)$$

Where T_1, T_2 is kinetic energy of particle in positions 1 and position 2 respectively.

Thus the work done by force acting on particle appears change in kinetic energy i.e. $W_{12} = \int_1^2 F.dr = T_2 - T_1 = change\ in\ kinetic\ energy$

This is known as work energy theorem.

If work done by the force in moving a particle from point 1 to point 2 is the same for any possible path between points, then force is conservative.

If the forces are derivable from scalar potential energy function V i.e. F = -∇V

Then total energy of particle is conserved. The work done is

$$W_{12} = \int_1^2 F.dr$$

$$= \int_1^2 -\nabla \mathrm{V}.dr = \int_1^2 -\frac{\mathrm{dV}}{\mathrm{dr}}.dr$$

$$= \int_1^2 -dV = V_1 - V_2 \quad \text{-------------------------} (3)$$

Which is change in potential energy when particle moves from point 1 to point 2.

From eq^n(2) and eq^n (3)

$$T_2 - T_1 = V_1 - V_2$$

$$\therefore T_1 + V_1 = T_2 + V_2 = \text{constant}$$

Thus sum of kinetic energy and potential energy of a particle remains constant in conservative force of field. This is law of conservation of energy.

2. **Mechanics of a system of particle:**

When the mechanical system consists of two or more particles, then we must distinguish between external force and internal force.

∴ Force on i^{th} particle is,

$$F_i = F_i^{(e)} + \sum_{j \neq i} F_{ij} \quad ------ (1)$$

Where $F_i^{(e)}$ is external force acting on i^{th} particle and F_{ij} is internal force on i^{th} particle due to j^{th} particle.

All the particles of system exert forces on one another hence internal force on i^{th} particle must be sum of forces due to all other particles excluding the term j=i, Since by definition the i^{th} particle on itself is zero.

According to Newton's second law of motion,

$$F_i = m_i a_i = \frac{dP_i}{dt}$$

$$= \frac{d}{dt}(m_i v_i) = m_i \frac{dv_i}{dt}$$

$$= m_i \frac{d}{dt}\left(\frac{dr_i}{dt}\right) = m_i \frac{d^2 r_i}{dt^2}$$

Now, when sum is taken over all the particles of system, eq^n(1) becomes

$$F_i = \frac{d^2}{dt^2} \sum_i m_i r_i = \sum F_i^{(e)} + \sum_{i,j} F_{ij} \quad ------ (2)$$

On right hand side of eq^n(2) first sum represents the total external force $F^{(e)}$. According to Newton's third law of motion, any two particles of system exert equal and opposite force on each other i.e. $F_{ij} + F_{ji} = 0$ --------- (3)

∴ eq^n (2) becomes,

$$\frac{d^2}{dt^2} \sum_i m_i r_i = \sum F_i^{(e)} \quad ------ (4)$$

Centre of mass:

We have to explain the concept of centre of mass of system of particles. Let R denotes position of centre of mass is defined as the average of radii vectors of particles weighted in proportion to their mass i.e.

$$R = \frac{\sum_i m_i r_i}{\sum m_i} = \frac{\sum_i m_i r_i}{M}$$

Where $M = \Sigma m_i$ is total mass of the system.

$\therefore$ eq^n (4) becomes,

$$M\frac{d^2R}{dt^2} = \sum_i F_i^{(e)} = F^{(e)}$$

$$\therefore F^{(e)} = M\frac{d^2R}{dt^2} = M\ddot{R}$$

i) **Conservation of linear momentum:**

We have position of centre of mass is

$$R = \frac{\sum_i m_i r_i}{M}$$

Differentiate with respect to t

$$M\frac{dR}{dt} = \frac{d}{dt}\left[\sum_i m_i r_i\right]$$

$$= \frac{d}{dt}[m_1 r_1 + m_2 r_2 \pm ----m_N r_N]$$

$$= m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + ----m_N\frac{dr_N}{dt}$$

$$= m_1 v_1 + m_2 v_2 + --- + m_N v_N$$

$$Mv = \sum_{i=1}^{N} m_i v_i$$

But $\Sigma m_i v_i = P$ is total linear momentum of particles of system.

$$\therefore P = Mv \text{ ------------------- } (5)$$

Differentiate eq^n (2) with respect to t

$$\frac{dP}{dt} = \frac{d}{dt}(Mv) = M\frac{dv}{dt} = M\frac{d^2R}{dt^2} \text{ ---------- (6)}$$

We have

$$F^{(e)} = M\frac{d^2R}{dt^2} = \frac{dP}{dt} = \frac{d}{dt}(Mv)$$

When $F^{(e)} = 0$,

$$\therefore P = Mv = \sum_{i=1}^{N} m_i v_i = constant$$

If total external force $F^{(e)}$ acting on the system of particle is zero, its total linear momentum is constant i.e. linear momentum is conserved.

ii) **Conservation of angular momentum:**

Let L_1, L_2, ------- L_N be angular momentum of N particles of a system.

Total angular momentum is

$L = L_1 + L_2 +$ -----$+ L_N$

$= (r_1 \times P_1) + ((r_2 \times P_2) +$ ------ $+ (r_N \times P_N)$

$L = \sum_{i=1}^{N}(r_i \times P_i)$------------------------- (7)

Differentiate with respect to t

$$\frac{dL}{dt} = \frac{d}{dt}\left[\sum_i (r_i \times P_i)\right] = \sum_i \left[r_i \times \frac{dP_i}{dt} + \frac{dr_i}{dt} \times P_i\right]$$

$$\frac{dL}{dt} = \sum_i \left[r_i \times \frac{dP_i}{dt} + v_i \times mv_i\right]$$

Where $v_i = dr_i / dt$ and $P_i = mv_i$

$But\ v_i \times mv_i = 0$

$$\frac{dL}{dt} = \sum_i \left[r_i \times \frac{dP_i}{dt}\right] = \sum_i [r_i \times F_i]$$

But

$$F_i = \sum F_i^{(e)} + \sum_{i,j} F_{ij}$$

$$\frac{dL}{dt} = \sum_i \left[r_i \times F_i^{(e)}\right] + \sum_{i,j} (r_i \times F_{ij})$$

Second term denotes the sum of internal torques which vanishes if the interacting forces are Newtonian.

$$\frac{dL}{dt} = \sum_i \left[r_i \times F_i^{(e)}\right] = \tau^{(e)}$$

If $\tau^{(e)} = 0$ then $dL/dt = 0$ and $L = L_1 + L_2 + \text{-----} + L_N$ = constant.

Thus in the absence of external torque, total angular momentum of system of particles is constant.

iii) Conservation of energy:

Total amount of work done by the forces acting on particles of the system from position 1 to position 2 is

$$W_{12} = \sum_{i=1}^{N} \int_1^2 F.dr = \sum_{i=1}^{N} \int_1^2 F_i^{(e)} dr_i + \sum_i \sum_j \int_1^2 F_{ij} dr_i \text{ -----(8)}$$

According to Newton's second law of motion

$$F_i = m_i \frac{dv_i}{dt}$$

$$\therefore W_{12} = \sum_{i=1}^{N} \int_1^2 F_i.dr_i = \sum_{i=1} \int_1^2 m_i \frac{dv_i}{dt}.\frac{dr_i}{dt} dt$$

$$= \sum_{i=1} \int_1^2 m_i \frac{dv_i}{dt}.v_i dt = \sum_{i=1} \int_1^2 \frac{d}{dt}\left[\frac{1}{2} m_i v^2{}_i\right] dt$$

$$= \sum_{i=1} \left[\frac{1}{2} m_i v^2{}_i\right]_1^2$$

But $\sum_i \frac{1}{2} m_i v_i{}^2 = T$ is kinetic energy of system of particle.

$W_{12} = [T]_1^2 = T_2 - T_1$------------------------- (9)

Thus the work done is equal to change in kinetic energy.

We know that, if the external and internal forces both are conservative and derivable from scalar potential then

$$F_i^{(e)} = -\nabla_i V_i \quad \text{and} \quad F_{ij} = -\nabla_i V_{ij}$$

∴

$F_i^{(e)} = -\nabla_i V_i \; = -\left[\vec{\imath}\frac{\partial V_i}{\partial x_i} + \vec{\jmath}\frac{\partial V_i}{\partial y_i} + \vec{K}\frac{\partial V_i}{\partial z_i}\right]$ ------ (10)

and

$F_{ij} = -\nabla_i V_{ij} = -\left[\vec{\imath}\frac{\partial V_{ij}}{\partial x_i} + \vec{\jmath}\frac{\partial V_{ij}}{\partial y_i} + \vec{K}\frac{\partial V_{ij}}{\partial z_i}\right]$ --------- (11)

If the internal forces are central in nature, the potential energy V_{ij} will be function of scalar distance $r_{ij} = |r_i - r_j|$ only. Then $V_{ij} = V_{ij}(|r_i - r_j|)$ -------------- (12)

So that

$$\frac{\partial V_{ij}}{\partial x_i} = \frac{\partial V_{ij}}{\partial r_{ij}}.\frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{x_{ij}}.\frac{\partial V_{ij}}{\partial r_{ij}} \quad ------ (13)$$

Since, $r_{ij} = |r_i - r_j| = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{1/2}$

Hence $\frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{x_{ij}}$

Similarly

$$\frac{\partial V_{ij}}{\partial y_i} = \frac{(y_i - y_j)}{y_{ij}}.\frac{\partial V_{ij}}{\partial r_{ij}}$$

and

$$\frac{\partial V_{ij}}{\partial z_i} = \frac{(z_i - z_j)}{z_{ij}}.\frac{\partial V_{ij}}{\partial r_{ij}}$$

∴ Eq^n (11) becomes,

$$F_{ij} = -\frac{1}{r_{ij}}.\frac{\partial V_{ij}}{\partial r_{ij}}\left[(x_i - x_j)\vec{I} + (y_i - y_j)\vec{J} + (z_i - z_j)\vec{K}\right]$$

$$= -(r_i - r_j)\frac{1}{r_{ij}}.\frac{\partial V_{ij}}{\partial r_{ij}}$$

Similarly,

$$F_{ji} = -\nabla_j V_{ij} = (r_i - r_j)\frac{1}{r_{ij}}.\frac{\partial V_{ij}}{\partial r_{ij}}$$

Thus the internal forces F_{ij} and F_{ji} between i^{th} and j^{th} particles are equal and opposite and vanish.

Consider the last term of eqn(8)

$$\sum_i \sum_j \int_1^2 F_{ij} dr_i = \frac{1}{2}\sum_i \sum_j \int_1^2 (F_{ij} dr_i + F_{ji} dr_j)$$

$$= \frac{1}{2}\sum_i \sum_j \int_1^2 -(\nabla_{\mathrm{i}} V_{ij} dr_i + \nabla_{\mathrm{i}} V_{ij} dr_j)$$

A factor ½ comes because of while summing the mutual potential energies, a pair of particles i, j appears a twice.

$$\sum_i \sum_j \int_1^2 F_{ij} dr_i = -\frac{1}{2}\sum_i \sum_j \int_1^2 -\nabla_{\mathrm{ij}} V_{ij} dr_{ij} \quad ------(14)$$

Therefore eq^n(8) becomes,

$$W_{12} = -\sum \int_1^2 \nabla V_i dr_i - \frac{1}{2}\sum_i \sum_j \int_1^2 -\nabla_{\mathrm{ij}} V_{ij} dr_{ij}$$

$$= -\sum \int_1^2 \frac{dV_i}{dr_i} dr_i - \frac{1}{2}\sum_i \sum_j \int_1^2 \frac{dV_{ij}}{dr_{ij}} dr_{ij}$$

$$= -\left[\sum_i [V_i]_1^2 + \sum_i \sum_j [V_{ij}]_1^2\right] = V_1 - V_2 \quad ------(15)$$

Where V is total potential energy of system is defined as

$$V = \sum_i V_i + \frac{1}{2}\sum_i \sum_j V_{ij}$$

From eq^n (9) and eq^n (15)

$$T_2 - T_1 = V_1 - V_2$$

$$\therefore T_1 + V_1 = T_2 + V_2 = \text{constant}$$

This is law of conservation of energy.

3. Constraints:

The restrictions of motion of a system of particle along the specified path are called as constraints and the motion is said to be constrained motion.

Here one coordinate is sufficient to describe the motion in contrast to the situation where the particle is free to move in space and three coordinates are needed to describe its motion. Thus imposing constraints on a mechanical system is to simplify the mathematical description. Constraints reduce the number of coordinates needed to specify the configuration of a system. When a particle(bead) is made to a slide on a wire constraint require that position of bead lie on wire. Condition imposed on the system by constraints can be written mathematically as a relation satisfied by the coordinates of particle at any time.

Example 1). Let us consider the motion of a simple pendulum confined to move in vertical plane. We need only two coordinates i.e. Cartesian coordinates x and y or polar coordinates r and θ to locate the position of the bob in motion. However motion of bob is not free but takes place under a constraint that the distance l of the bob is to remain same at all time. This condition imposed by the constraint can be expressed in form of equation either between x and y or r and θ such as, $x^2 + y^2 = l^2$ or $r = l$

Again, in polar coordinate equation looks simple. One coordinates either x and y in cartesian coordinates or θ in polar coordinate should sufficient to describe motion.

2). Suppose a particle moving in space requires three coordinates to determine its position at any instant. If we restrict its movement on the surface of sphere, there exists a relation between these coordinates. Again we shall see that spherical polar coordinates can be used with advantage as,

$x^2 + y^2 + z^2 = a^2$ or $r = a$ where a is radius of sphere.

Types of Constraints:

There are two types of constraints:

i) Holonomic constraints
ii) Non-Holonomic constraints

i). Holonomic constraints: If we express the conditions of constraints as equations connecting the coordinates of particle having the form $f(r_1, r_2, \text{-----} t) = 0$, then the constraints are said to be holonomic. i.e. holonomic constraints depends only on the coordinates and time.

Ex. 1. The constraints of rigid body is defined as the one where distance between any two particle remains constant during the motion may be given by,

$$(r_i - r_j)^2 - C_{ij}^2 = 0$$

Where C_{ij} is distance between particles i and j which is at position r_i and r_j.

2. A particle constrained to move on an inclined plane must have x and y components to satisfy condition, $\emptyset(x, y, z, t) = y - xtan\theta = 0$

Where x and y are measured from inclined plane.

3. Suppose a particle is moving on rim of circle of radius a. Here we obtain the rim Y-Z plane where the centre of circle and origin coincide. In this, the constraints is that particle remains on rim. The constraints equation is

$$x = 0 \; and \; y^2 + z^2 = a^2$$

ii). Non-Holonomic constraints: The constraints which cannot be expressed in the form of equations are non-holonomic constraints.

If constraint is non-holonomic, the equation expressing the constraint cannot be used to eliminate the dependent coordinates. An example of non-holonomic constraint is that an object rolling on surface without slipping. The coordinates used to describe the system will generally involve the angular coordinates to specify the orientation of the body plus the set of coordinates describing the location of point of contact on the surface.

Ex.1. The motion of particle placed on the surface of sphere under the action of gravitational force is bound by non-holonomic constraints and it can be expressed as an equality $r^2 - a^2 \geq 0$.

Equality sign holds until the particle rolls on sphere and when it leaves the sphere, we must have $r^2 - a^2 > 0$.

2. Consider a disc rolling on horizontal XY plane constrained to move so that the plane of disc is always vertical. The coordinates used to describe the motion might be the x, y coordinates of the centre of disc, an angle of rotation ϕ about the axis of the disc and an angle θ between the axis of the disc and say X-axis as shown in Fig. As a result of the constraints the velocity of the centre of the disc v has a magnitude proportional to $\dot{\emptyset}$, $v = a\dot{\emptyset}$

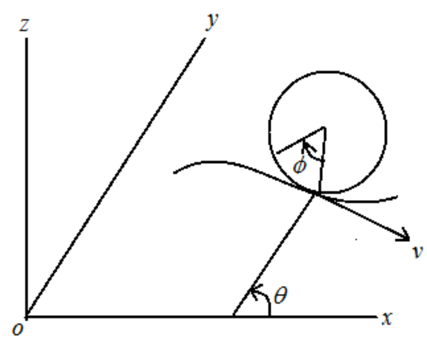


Figure 1 Vertical disc rolling on horizontal plane.

Where a is the radius of the disc, and its direction is perpendicular to the axis of the disc;

$$\dot{x} = v sin\theta \text{ and } \dot{y} = -v cos\theta$$

Combining these conditions we have two differential equations of constraint

$$dx - asin\theta d\emptyset = 0$$

$$dy + acos\theta d\emptyset = 0$$

This equation cannot be integrated without in fact solving the problem i.e. we cannot find an integrating factor $f(x, y, \theta, \phi)$ that will turn either of the equations into perfect differentials. Hence the constraints cannot be reduced to equation form and are non-holonomic.

Note: Constraints are further classified according to whether the equations of constraint contain the time explicit variable which is rheonomous or are not explicitly dependent on time which is scleronomous. A bead sliding on a rigid curved wire fixed in space is obviously subject to a scleronomous constraint; if the wire is moving in some prescribed fashion, the constraint is rheonomous.

4. Virtual Work:

A virtual displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates δr_i, consistent with the forces and constraints imposed on the system at the given instant *t*. There is no actual displacement during which forces and constraints may change and hence this displacement is called as virtual displacement. It is an imaginary displacement.

Work done by the force on particle when a virtual displacement is given to particle is called virtual work.

In this, the system is subjected to an infinitesimal displacement consistent with forces and constraints imposed on system at the given instant *t*.

Let δr_i be infinitesimal virtual displacement of i^{th} particle. Suppose system is in equilibrium i.e. total force F_i on every particle is zero, then work done by this force in a small virtual displacement δr_i will also zero.

$$\delta W_i = F_i.\delta r_i = 0$$

Similarly sum of virtual work for all particles must be zero i.e.

$$\delta W_i = \sum_i F_i.\delta r_i = 0 ------(1)$$

The total force be expressed as sum of applied force $F_i^{(a)}$ and forces of constraints f_i as

$$F_i = F_i^{(a)} + f_i$$

Therefore, eq^n (1) becomes,

$$\sum_i F_i^{(a)}.\delta r_i + \sum_i f_i.\delta r_i = 0 \; -------(2)$$

We restrict ourselves to systems for which the net virtual work of the forces of constraints is zero. We have seen that this condition holds true for rigid bodies and it is valid for a large number of other constraints. Thus, if particle is constrained to move on a surface, the force of constraint is perpendicular to the surface while the virtual displacement must be tangent to it and hence the virtual work vanishes.

The condition for equilibrium of a system that the virtual work of the applied forces vanishes is,

$$\sum_i F_i^{(a)}.\delta r_i = 0 \; -------(3)$$

This equation is called as the principle of virtual work.

5. D'Alembert's Principle:

According to Newton's second law of motion, force acting on ith particle is

$$F_i = \frac{dP_i}{dt} = \dot{P} \qquad \text{------------------} \; (1)$$

To interpret the equilibrium of systems, D'Alembert adopted an idea of a reversed force. He explained that a system will remain in equilibrium under the action of force equal to actual force Fi plus a reversed effective force$\dot{P_i}$. Thus

$$F_i + (-\dot{P_i}) = 0$$

$$F_i - \dot{P_i} = 0$$

The principle of virtual work takes the form

$$\sum_i (F_i - \dot{P_i}).\delta r_i = 0$$

Again writing

$$F_i = F_i^{(a)} + f_i$$

$$\sum_i (F_i^{(a)} - \dot{P_i}).\delta r_i + \sum_i f_i.\delta r_i = 0$$

Again we restrict with the systems for which virtual work of the forces of constraints is zero. Therefore,

$$\sum_i (F_i^{(a)} - \dot{P_i}).\delta r_i = 0$$

Since force of constraints do not appear in equation and hence we drop superscript 'a'. Therefore above equation becomes,

$$\sum_i (F_i - \dot{P_i}).\delta r_i = 0$$

This equation is called as D'Alembert principle.

6. Lagrange's Equation:

Consider a system of N particles. The coordinate transformation equations are

$$r_i = r_i(q_1, q_2, ---- q_n, t) \text{ -------------------- } (1)$$

Where q_1, q_2------- q_n are generalised coordinates.

Differentiate eq^n(1) with respect to t

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}.\frac{\partial q_1}{\partial t} + \frac{\partial r_i}{\partial q_2}.\frac{\partial q_2}{\partial t} + ------\frac{\partial r_i}{\partial t}.\frac{dt}{dt}$$

Velocity of ith particle is

$$v_i = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \quad ----------- (2)$$

Where $\dot{q}_j$ are generalised velocities

The virtual displacement is given by

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + ----+\frac{\partial r_i}{\partial q_n}\delta q_n + \frac{\partial r_i}{\partial t}\delta t$$

$$\delta r_i = \sum_i \frac{\partial r_i}{\partial q_j}\delta q_j + \frac{\partial r_i}{\partial t}\delta t$$

But last term is zero since virtual displacement only coordinate displacement and not that of time i.e. $\frac{\partial r_i}{\partial t}\delta t = 0$

$$\delta r_i = \sum_i \frac{\partial r_i}{\partial q_j}\delta q_j$$

According to D'Alembert's principle,

$$\sum_i (F_i - \dot{P}_i).\delta r_i = 0$$

$$\sum_i (F_i - \dot{P_i}).\frac{\partial r_i}{\partial q_j}\delta q_j = 0$$

$$\therefore \sum_{i,j} F_i \frac{\partial r_i}{\partial q_j}\delta q_j - \sum_{i,j} \dot{P_i}\frac{\partial r_i}{\partial q_j}\delta q_j = 0 \quad ------------- (3)$$

We define, $\sum_{ij} F_i \frac{\partial r_i}{\partial q_j} = Q_j$

are called the components of generalised force associated with generalised coordinates q_j.

∴ eqn (3) becomes,

$$\sum_j Q_j.\delta q_j - \sum_{i,j} \dot{P_i}\frac{\partial r_i}{\partial q_j}\delta q_j = 0 \quad ------------- (4)$$

$$\therefore \sum_{i,j} \dot{P_i}\frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{i,j} m_i \ddot{r_i}\ \frac{\partial r_i}{\partial q_j}\delta q_j$$

$$= \sum_{ij}\left[\frac{d}{dt}\left(m_i \dot{r_i}\ \frac{\partial r_i}{\partial q_j}\right) - m_i \dot{r_i}\ \frac{d}{dt}(\frac{\partial r_i}{\partial q_j})\right]\delta q_j$$

$$\left\{since, \frac{d}{dt}\left(m_i \dot{r_i}\ \frac{\partial r_i}{\partial q_j}\right) = m_i \ddot{r_i}\ \frac{\partial r_i}{\partial q_j} + m_i \dot{r_i}\ \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right\}$$

$$\left\{m_i \ddot{r_i}\ \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left(m_i \dot{r_i}\ \frac{\partial r_i}{\partial q_j}\right) - m_i \dot{r_i}\ \frac{d}{dt}(\frac{\partial r_i}{\partial q_j})\right\}$$

$$\therefore \sum_{i,j} \dot{P_i}\frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{ij}\left[\frac{d}{dt}\left(m_i v_i \frac{\partial r_i}{\partial q_j}\right) - m_i v_i \frac{d}{dt}(\frac{\partial r_i}{\partial q_j})\right]\delta q_j \quad --- (5)$$

Further, $\quad \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{dr_i}{dt}\right) = \frac{\partial v_i}{\partial q_j}$

Differentiate eqn(2) with respect to $\dot{q}_j$

$$\therefore \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

$\therefore$ eqn (5) becomes

$$\therefore \sum_{i,j} \dot{P}_i \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{ij} \left[\frac{d}{dt} \left(m_i v_i \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i \frac{\partial v_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} (T) - \frac{\partial}{\partial q_j} (T) \right\} \right] \delta q_j$$

Where $\sum_i \frac{1}{2} m_i v_i^2 = T$ is kinetic energy of system.

By substituting these values in eqn (4), we have

$$\sum_j Q_j . \delta q_j - \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} (T) - \frac{\partial}{\partial q_j} (T) \right\} \right] \delta q_j = 0$$

$$\therefore \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

Since the constraints are holonomic, q_j are independent of each other and hence coefficient of δq_j vanish i.e.

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] = Q_j \quad ------(6)$$

Eqn (6) is general form of Lagrange's equation.

Case I (For conservative system):

For conservative system, forces are derivable from potential function V,

$$\therefore F_i = -\nabla V_i = -\frac{\partial V}{\partial r_i}$$

The generalised force can be expressed as

$$Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j} = -\sum_i \nabla V_i \frac{\partial r_i}{\partial q_j}$$

$$Q_j = -\sum_i \frac{\partial V}{\partial r_i}\frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

The general form of Lagrange's equation becomes

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = -\frac{\partial V}{\partial q_j}$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j}\right] = 0$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j}\right] = 0$$

$$\left[\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j}\right] = 0$$

Since V is not function of $\dot{q}_j$.

We define new function given by

L= T-V called as Lagrangian for the conservative system.

∴ above equation becomes,

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right]=0$$

This equation is known as Lagrange's equation of motion for conservative system.

Case II (For non-conservative system): If potentials are velocity dependent, called as generalised potential, then system is not conservative. Then we obtain generalised force as function U (qj, $\dot{q}_j$) such that

$$Q_j=-\frac{\partial U}{\partial q_j}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

Then Lagrange's equation becomes,

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}\right]=Q_j$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}\right]=-\frac{\partial U}{\partial q_j}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial U}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial U}{\partial q_j}=0$$

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right)-\frac{\partial (T-U)}{\partial q_j}=0$$

If we take Lagrangian L= T-U where U is generalised potential.

Above equation becomes,

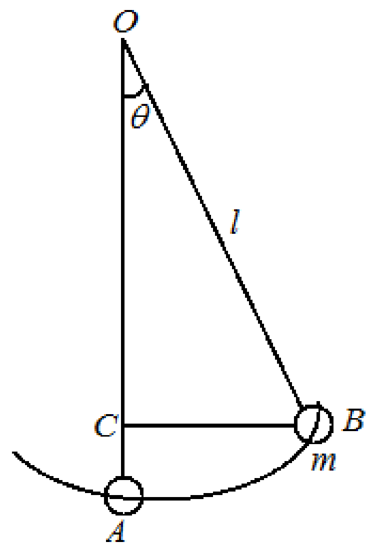
$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right]=0$$

This is Lagrange's equation of motion.

7. Simple Applications of Lagrangian Formulation:-

i) Simple Pendulum:

Simple pendulum is a point mass suspended by a light, inextensible string from a rigid support.



Suppose l is length of pendulum.

θ be angle through which the pendulum is displaced from its equilibrium position and it si chosen as generalised coordinates.

Kinetic energy of pendulum is,

$$T = \frac{1}{2}mv^2$$

$$But\ v = lw = l.\frac{d\theta}{dt} = l\dot{\theta}$$

$$\therefore Kinetic\ energy\ T = \frac{1}{2}ml^2\dot{\theta}^2$$

In coming from position B to A, the mass has fallen freely through a vertical distance CA.

∴ Potential energy is,

$$V = mg.CA = mg(OA - OC)$$

But $OA = l$ and $OC = OB\ cos\theta = lcos\theta$

$$V = mg\ (l - lcos\theta) = mgl(1 - cos\theta)$$

We know that, the Lagrangian is, $L = T - V$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - cos\theta) - - - - - (1)$$

Now Lagrangian equation of motion is,

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}\right] = 0 \; ------ (2)$$

Differentiate eq^n(1) with respect to $\dot{\theta}$ and θ, we get

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} ml^2 . 2\dot{\theta} = ml^2\dot{\theta}$$

And $\quad \frac{\partial L}{\partial \theta} = 0 - mgl\,(0 + sin\theta) = -mglsin\theta$

Putting these values in eq^n (2),

$$\therefore \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mglsin\theta = 0$$

$$ml^2\ddot{\theta} + mglsin\theta = 0$$

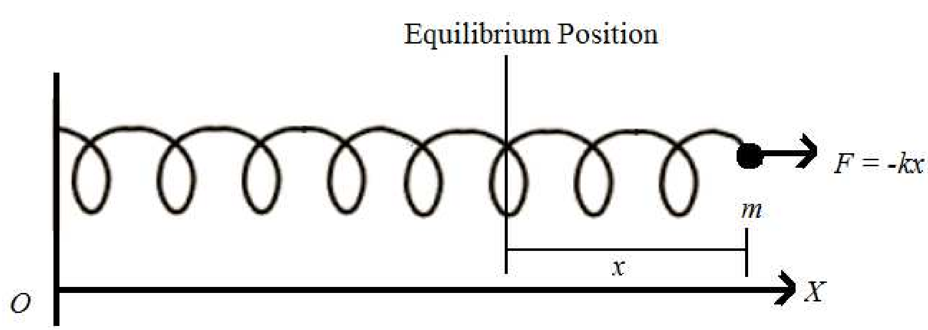
$$\ddot{\theta} + \frac{g}{l} sin\theta = 0 ------ (3)$$

This is required equation of motion for simple pendulum.

If amplitude of motion is small then $\sin\theta \approx \theta$.

$$\therefore \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$

ii) Linear Harmonic Oscillator:



Let us consider the motion in direction of X-axis. The kinetic energy of harmonic oscillator is given by

$$T = \frac{1}{2}mv^2$$

$$But\ v = \frac{dx}{dt} = \dot{x}\ \ be \text{ velocity of particle}$$

$$\therefore T = \frac{1}{2}m\dot{x}^2$$

The potential energy of particle is, $V = -\int F.dx$

Where $F = -kx$ is restoring forces acting on particle and k is force constant.

$$V = -\int -kx.dx = k.\frac{x^2}{2} + C$$

Where C is constant of integration. If we choose horizontal plane passing through the position of equilibrium as reference level then v = 0 at x = 0. So that C = 0.

$$\therefore Potential\ energy\ V = \frac{kx^2}{2}$$

We know that, the Lagrangian is,

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2} \quad ------(1)$$

Now Lagrangian equation of motion is,

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}\right] = 0\ ------(2)$$

Differentiate eq^n(1) with respect to $\dot{x}$ and x, we get

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m.2\dot{x} = m\dot{x}$$

$$\frac{\partial L}{\partial x} = 0 - \frac{1}{2}k.2x = -kx$$

Putting these values in eq^n (2),

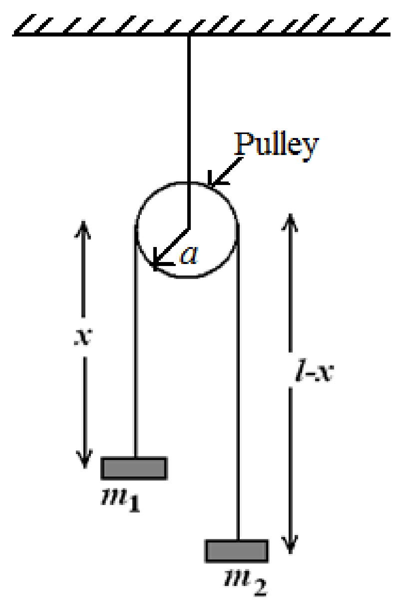
$$\left[\frac{d}{dt}(m\dot{x}) + kx\right] = 0$$

$$\therefore m\ddot{x} + kx = 0 - - - - (3)$$

Eq^n (3) is required equation of motion of one dimensional harmonic oscillator.

iii) Atwood's Machine:

Atwood's machine is holonomic conservative system.



System consists of two masses m_1 and m_2 suspended over a frictionless pulley of radius *a* and connected by a flexible string of constant length *l*.

Suppose *x* is variable particle distance from pulley to mass m_1 then mass m_2 is at distance *l-x* from pulley.

There is only one independent coordinate *x*. The velocities of two masses are

$$v_1 = \frac{dx}{dt} = \dot{x} \quad and \; v_2 = \frac{d(l-x)}{dt} = -\dot{x}$$

Kinetic energy of the system is

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

The potential energy of the system is

$$V = -m_1 g x - m_2 g(l - x)$$

We know that, the Lagrangian is,

$$L = T - V$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g(l - x) \quad ------(1)$$

Now Lagrangian equation of motion is,

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}\right] = 0 \;\; ------(2)$$

Differentiate eq^n(1) with respect to $\dot{x}$ and x, we get

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}(m_1 + m_2).2\dot{x} = (m_1 + m_2)\dot{x}$$

$$\frac{\partial L}{\partial x} = m_1 g + m_2 g(-1) = (m_1 - m_2)g$$

Putting these values in eq^n (2),

$$\frac{d}{dt}[(m_1 + m_2)\dot{x}] - (m_1 - m_2)g = 0$$

$$(m_1 + m_2)\ddot{x} - (m_1 - m_2)g = 0$$

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$\ddot{x} = \frac{(m_1 - m_2)g}{(m_1 + m_2)} \;\; ------(3)$$

This is required equation of motion of Atwood's machine.

Multiple Choice Question:

1. The number of independent variable for a free particle in space are ______

(a) one (b) two **(c) three** (d) zero

Ans: c

2. ___________ constraints are independent of time.

(a) Holonomic (b) Non-Holonomic

(c) Scleronomous (d) Rheonomous

Ans: c

3. ___________ constraints are time dependent.

(a) Holonomic (b) Non-Holonomic

(c) Scleronomous **(d) Rheonomous**

Ans: d

4. The Lagrangian equations of motion are ___________ order differential equations.

(a) first (b) **second** (c) zero (d) forth

Ans: b

5. If the total external force acting on the body is zero, which of the following physical quantity is conserved?

(a) force **(b) linear momentum**

(c) angular momentum (d) energy

Ans: b

6. If the external torque acting on the body is zero, which of the following physical quantity is conserved?

(a) force (b) linear momentum

(c) angular momentum (d) energy

Ans: c

7. If work done by the force in moving a particle from point 1 to point 2 is the same for any possible path between points, then force is --------.

(a) conservative (b) non-conservative

(c) Both (a) and (b) (d) none of these

Ans: a

8. If the forces *F* are derivable from scalar potential energy function *V* then relation between force and potential energy is ---

(a) $F = -\nabla V$ (b) $F = \nabla V$ (c) $V = -\nabla F$ (d) $V = \nabla F$

Ans: a

(9) Which of the following correct statement about constraints?

(a) It is the restrictions of motion of a system of particle along the specified path.

(b) It reduces the number of coordinates needed to specify the configuration of a system.

(c) Condition imposed by it can be written mathematically as a relation satisfied by the coordinates of particle at any time.

(d) All of these

Ans: d

(10) The constraints which cannot be expressed in the form of equations are ---.

(a) Holonomic **(b) Non-holonomic**

(c) generalised (d) conservative

Ans: b

(11) The constraints which can be expressed in the form of equations are ---.

(a) Holonomic (b) Non-holonomic

(c) generalised (d) conservative

Ans: a

(12) The equation of the principle of virtual work is -----

(a) $\boldsymbol{\sum F_i{}^{(a)}.\delta r_i = 0}$ (b) $\sum(F_i - \dot{P}_i).\delta r_i = 0$

(c) F = -∇V (d) $\sum(F_i + \dot{P}_i).\delta r_i = 0$

Ans: a

(13) The equation of the D'Alembert principle is -----

(a) $\sum F_i{}^{(a)}.\delta r_i = 0$ **(b)** $\boldsymbol{\sum(F_i - \dot{P}_i).\delta r_i = 0}$

(c) F = -∇V (d) $\sum(F_i + \dot{P}_i).\delta r_i = 0$

Ans: b

(13) For non-conservative system, potential depends on -------

(a) position (b) force (c) momentum **(d) velocity**

Ans: d

(14) General Lagrange's equation of motion is ------

(a) $\left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j$ (b) $\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right] = Q_j$

(c) $\boldsymbol{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j}$ (d) $\left[\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right)\right] = Q_j$

Ans: c

(15) Differential equation of motion for simple pendulum is ------

(a) $\boldsymbol{\ddot{\theta} + \frac{g}{l} sin\theta = 0}$ (b) $\theta + \frac{g}{l} sin\theta = 0$

(c) $\ddot{\theta} + \frac{l}{g} sin\theta = 0$ (d) $\ddot{\theta} + \frac{g}{l} cos\theta = 0$

Ans: a

(16) For simple pendulum, the Lagrangian function *L* is written as -------

(a) $L = \frac{1}{2} ml^2\dot{\theta}^2 + mgl(1 - cos\theta)$ **(b)** $\boldsymbol{L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1 - cos\theta)}$

(c) $L = \frac{1}{2} ml^2\theta - mgl(1 - cos\theta)$ (d) $L = \frac{1}{2} ml^2\theta^3 - mgl(1 - cos\theta)$

Ans: b

(17) Differential equation of motion of harmonic oscillator is --------

(a) $m\dot{x} + kx = 0$ (b) $m\ddot{x} - kx = 0$

(c) $mx + k\ddot{x} = 0$ **(d)** $\boldsymbol{m\ddot{x} + kx = 0}$

Ans: d

(18) For harmonic oscillator, the Lagrangian function L is written as----

(a) $L = \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2}$

(b) $L = \frac{1}{2}m\ddot{x}^2 - \frac{kx^2}{2}$

(c) $L = \frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2}$

(d) $L = \frac{1}{2}mx^2 - \frac{k\ddot{x}^2}{2}$

Ans: c

(19) Atwood's machine is ---------- conservative system.

(a) Holonomic

(b) Non-Holonomic

(c) Scleronomous

(d) Rheonomous

Ans: a

(20) Differential equation of motion of Atwood's machine is -------

(a) $\ddot{x} = \frac{(m_1+m_2)g}{(m_1+m_2)}$

(b) $\ddot{x} = \frac{(m_1-m_2)g}{(m_1+m_2)}$

(c) $\ddot{x} = \frac{(m_1+m_2)g}{(m_1-m_2)}$

(d) $\dot{x} = \frac{(m_1-m_2)g}{(m_1+m_2)}$

Ans: b